FINITE GROUPS IN WHICH EVERY PROPER NORMAL SUBGROUP IS CYCLIC

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Abstract. Let G be a finite non-simple, non-cyclic group with the property that all proper normal subgroups of G are cyclic. We call such a group NCS group, short for *Normal Cyclic Subgroups*. In this paper, after presenting some basic properties of these groups, we provide a complete classification. In particular, we show that an NCS groups is either the non-cyclic group of order p^2, with p prime, or Q_8, or a particular Z-group, or is a cyclic extension of a finite simple group.

1. Introduction

Let $\mathfrak{C}$ be a class of groups. A finite group G is said to be $\mathfrak{C}$-*critical*, or *minimal non*-$\mathfrak{C}$, if G itself does not belong to $\mathfrak{C}$, but all of its proper subgroups do. For example, a minimal non-cyclic group is a finite non-cyclic group in which all proper subgroups are cyclic. Many authors have investigated these groups for some classes $\mathfrak{C}$, with the goal of achieving a complete classification of *minimal non*-$\mathfrak{C}$. The first significant result in this area was obtained in [10], where the authors have classified minimal non-abelian groups and minimal non-cyclic groups. Later, classifications have been achieved also for minimal non-nilpotent groups (including some generalizations of these), and for minimal non-supersolvable groups (see [4], [8], [3] respectively).
Alongside, some authors have explored groups containing a specific family of subgroups within $\mathfrak{C}$. The most notable example in this sense is the study of groups in which all Sylow p-subgroups are cyclic. These groups, introduced by Suzuki in [13] and known as Z-groups, will play a key role in the proof of our main result. Z-groups have been fully classified, see Section 3 below.

In this paper, we study groups in which all normal subgroups belong to a certain class of groups $\mathfrak{C}$.
Let G be a finite non-simple group and let $\mathcal{N}(G)$ denote the set of all proper normal subgroups of G. If G is non-cyclic and all the elements of $\mathcal{N}(G)$ are cyclic, we say that G is an NCS group, short for *Normal Cyclic Subgroups*.
The smallest example of this kind of group is S_3, the symmetric group of degree 3, which has $A_3 = C_3$ as its unique normal subgroup. This example is somewhat trivial since all of its subgroups are cyclic. Naturally, if G is a minimal non-cyclic

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group, then G is also an NCS group. However, the converse is not true. The smallest example illustrating this is the dihedral group D_{18} of order 18. All of its normal subgroups are cyclic, yet it contains S_3 as a subgroup, which is not cyclic. Our main result provides a complete classification of NCS groups.

Theorem 1.1. *Let G be a finite group. Then, G is an* NCS *group if, and only if, G is one of the following.*

(1.1) *$G \cong C_p \times C_p$, where p is a prime.*
(1.2) *$G = Q_8$, the quaternion group.*
(1.3) *G is a group with the following presentation*

$$G = \langle a, b \,|\, a^m = b^{p_1^{\alpha_1}} = 1,\, bab^{-1} = a^r \rangle,$$

with $m = p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}$, $t > 0$, $p_1 < p_2 < \cdots < p_n$ primes, α_i integers for $i = 1, \ldots, t$ and

$$(m, p_1^{\alpha_1}) = (m, r-1) = 1,\, r^{p_1} \equiv 1 \mod m.$$

(1.4) *G is a perfect group with a unique maximal normal subgroup, which fits in a short exact sequence of the form*

$$1 \to C \to G \to S \to 1,$$

where S is a simple group and C is cyclic. Moreover, all possibilities for S and C are listed in Table 1.

Observe that among the groups appearing in Theorem 1.1, those of type 1.1 and Q_8 are minimal non-cyclic. A group of type 1.3 is minimal non-cyclic if and only if $m = q$ for some prime q, while a group of type 1.4 is never minimal non-cyclic.

Groups of this type have already been classified in [5] (in Italian). Our approach here is completely independent and different. Furthermore, we are able to provide additional insights compared to [5] in the case of perfect groups. Consequently, we manage to complete the previous classification with a description of all the possibilities for the simple group S and the normal cyclic group C of the short exact sequence appearing in the case 1.4 of Theorem 1.1.

The paper is organized as follows. In Section 2, we present some basic properties of NCS groups. Additionally, we recall the definition and some properties of supersolvable groups, which will be used in the proof of Theorem 1.1. We then proceed with the proof of Theorem 1.1. The proof is divided in two cases. In Section 3, we address the case where G' is a proper subgroup of G, while in Section 4, we handle the case where G is perfect.

2. Preliminaries

In this section, we show some basic and generic results about NCS groups.

Lemma 2.1. *Let G be an* NCS *group. Then, every non-simple quotient of G is either cyclic or an* NCS *group.*

Proof. Take $N \trianglelefteq G$. Suppose that N is not maximal, and that G/N is not cyclic. Take $H/N \trianglelefteq G/N$. Then, $N \trianglelefteq H \trianglelefteq G$. Since H is cyclic, H/N must also be cyclic. □

Simple group S	C is a non-trivial subgroup of	Comments
A_n	C_2	$n = 5$ or $n \geq 8$
A_n	C_6	$n = 6, 7$
$\mathrm{PSL}_n(q)$	$C_{(n,q-1)}$	$(n, q) \neq (2, 4), (2, 9), (3, 2), (3, 4)$, $(n, q-1) \neq 1$.
$\mathrm{PSL}_2(4)$	C_2	
$\mathrm{PSL}_2(9)$	C_6	
$\mathrm{PSL}_3(2)$	C_2	
$\mathrm{PSL}_3(4)$	C_{12}	
$\mathrm{PSp}_{2n}(q)$	C_2	q odd, $(n, q) \neq (3, 2)$
$\mathrm{PSp}_6(2)$	C_2	
$\mathrm{PSU}_n(q)$	$C_{(n,q+1)}$	$(n, q) \neq (4, 2), (4, 3), (6, 2)$ $(n, q+1) \neq 1$
$\mathrm{PSU}_4(2)$	C_2	
$\mathrm{PSU}_4(3)$	C_{12}	
$\mathrm{PSU}_6(2)$	C_6	
$\Omega_{2n+1}(q)$	C_2	$(n, q) \neq (3, 2), (3, 3)$ q odd.
$\Omega_7(2)$	C_2	
$\Omega_7(3)$	C_6	
$\Omega^+_{2n}(q)$	C_2	$(n, q) \neq (4, 2)$, n even, $(4, q^n - 1) \neq 1$
$\Omega^+_8(2)$	C_2	
$\Omega^+_{2n}(q)$	$C_{(4,q^n-1)}$	n odd, $(4, q^n - 1) \neq 1$
$\Omega^-_{2n}(q)$	$C_{(4,q^n-1)}$	$(4, q^n - 1) \neq 1$
$E_6(q)$	C_3	$(3, q-1) \neq 1$
$E_7(q)$	C_2	$(2, q-1) \neq 1$
$F_4(2)$	C_2	
$G_2(3)$	C_3	
$G_2(4)$	C_2	
${}^2E_6(q^2)$	C_3	$q \neq 2$, $(3, q+1) \neq 1$
${}^2E_6(4)$	C_6	
${}^2B_2(8)$	C_4	
M_{12}	C_2	
M_{22}	C_{12}	
J_n	C_n	$n = 2, 3$
Co_1	C_2	
Fi_{22}	C_6	
Fi'_{24}	C_3	
HS	C_2	
McL	C_3	
Ru	C_2	
Suz	C_6	
O′N	C_3	
B	C_2	

TABLE 1. NCS quasisimple groups

The NCS property does not necessarily pass to subgroups. For example, consider $G = \mathrm{SL}_2(7)$. G has a unique proper normal subgroup, which is its center, and this is cyclic, so G is an NCS group. However, G has a 2-Sylow subgroup isomorphic to the generalized quaternion group of order 16, Q_{16}. The group Q_{16}, in turn, has the quaternion group Q_8 as normal subgroup, which is not cyclic.

Recall now that a group G is said to be supersolvable if it has a normal series with cyclic quotients between consecutive terms, that is a series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G,$$

where $G_i \trianglelefteq G$ and G_i/G_{i-1} is cyclic, for all $i = 1, \dots, n$.
We need two basic facts about supersolvable groups.

Lemma 2.2. [1, Lemma 2.17] *Let G be a finite group with G' cyclic, where $G' = [G, G]$ is the commutator subgroup. Then G is supersolvable.*

Lemma 2.3. [6, Theorem 4.24] *Let G be a finite supersolvable group, and let p be the smallest prime dividing the order of G. Then, the elements of order prime to p form a normal π-Hall subgroup of G, where π is the set of primes dividing the order of G different from p.*

A group G is said to be a Z-group if all of its Sylow subgroups are cyclic. As we already said, Z-groups have been completely classified. In particular, we have the following theorem.

Theorem 2.4. [9, Theorem 9.4.3] *Let G be a Z-group. Then,*

$$G = \langle a, b \mid a^m = b^n = 1, bab^{-1} = a^r \rangle,$$

with $(m, n) = (m, r-1) = 1$, and $r^n \equiv 1 \mod m$. Moreover, $m = |G'|$ and $n = |G/G'|$.

The subgroups and normal subgroups of these groups are well known, and are described by the following theorem.

Theorem 2.5. [14, Theorem 1] *Let G be a Z-group with presentation as above. The normal subgroups of G are the subgroups of the form*

$$N = \langle a^{m_1}, b^{n_1} \rangle,$$

with $m_1 | (m, r^{n_1} - 1)$, $n_1 | n$.

3. Proof of Theorem 1.1: Case $G' < G$

We start with the abelian case, that is, $G' = 1$.

Proposition 3.1. *Let G be a finite abelian group. Then, G is an* NCS *group if, and only if, $G = C_p \times C_p$ for some prime p.*

Proof. The group $C_p \times C_p$ is trivially an NCS group.
Suppose that G is a finite abelian NCS group. Since G is abelian, we have $G = C_{p_1^{\alpha_1}} \times \cdots \times C_{p_t^{\alpha_t}}$, for some primes $p_1, \dots, p_t$ (not necessary distinct), and some integers $\alpha_1, \dots, \alpha_t$, and with $t \geq 2$. Suppose that $t \geq 3$. If the primes are pairwise distinct, then G is cyclic. Thus, there exist two indexes i, j such that $p_i = p_j$. This implies that $C_{p_i} \times C_{p_i}$ is a non-cyclic normal subgroup of G, which is impossible. Thus, $t = 2$, and $G = C_{p^\alpha} \times C_{p^\beta}$. Suppose that one between α and β is greater then

1. Then, $C_p \times C_p$ is a non-cyclic normal subgroup of G, which is again impossible. Hence $\alpha = \beta = 1$, and $G = C_p \times C_p$. □

We now move to the non-abelian case. To start with, we analyze the case where the group is a p-group.

Lemma 3.2. *Let G be a non-abelian* NCS *p-group, for some prime p. Then, the following hold.*

(3.1) *Every subgroup of G is cyclic.*
(3.2) *G admits a unique subgroup of order p.*

Proof.

(3.1) Since G is a p-group, every maximal subgroup of G is normal. In particular, every maximal subgroup of G is cyclic. But every subgroup is contained in a maximal subgroup, so every subgroup is cyclic.
(3.2) Consider the center of G, which is a non-trivial cyclic subgroup of G, and take a subgroup H of the center of order p. Since the center is characteristic in G, $H \trianglelefteq G$. Suppose now that K is another subgroup of G of order p. Thus, $HK \leq G$ is a cyclic subgroup of G and $|HK| \leq p^2$. Thus, H and K are two subgroups of order p of a cyclic group, implying that $H = K$.

□

Proposition 3.3. *Let G be a non-abelian p-group, for some prime p. Then, G is an* NCS *group if, and only, if $G = Q_8$.*

Proof. The proper normal subgroups of Q_8 are isomorphic either to C_4 or to C_2, so Q_8 is an NCS group.
Suppose now that G is a non-abelian NCS p-group. By Lemma 3.2, every subgroup of G is cyclic, and G admits a unique subgroup of order p. By [12, Theorem 9.7.3], $G = Q_{2^n}$, the generalized quaternion group, with $n > 2$. Aiming for a contradiction, suppose that $n \geq 4$. Thus, since $2^{n-2} \geq 4$, the dihedral group $D_{2^{n-2}}$ contains a proper non-cyclic subgroup. But $G/Z(G) = D_{2^{n-2}}$. Thus, also G admits a non-cyclic subgroup, which is a contradiction. Thus $n = 3$ and $G = Q_8$. □

From now on, we may suppose that G is a non-abelian NCS group, and that G is not a p-group. Recall that a group G is called Z-group if all of its Sylow p-subgroups are cyclic.

Proposition 3.4. *If G is an* NCS *group, which is not a p-group, and $1 \neq G' < G$, then G is a* Z*-group.*

Proof. Since G' is normal in G, G' is cyclic. By Lemma 2.2, G is supersolvable. Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, where $p_1 < p_2 < \cdots < p_t$ are distinct primes, α_i are non-negative integers and $t > 1$. Thus, by Lemma 2.3, G has a normal π-Hall subgroup N of order $m = p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, where $\pi = \{p_2, \ldots, p_t\}$. In particular, $N = C_m$. Now, if P is a Sylow p-subgroup of G, with $p \in \pi$, then $P \leq C_m$, hence P is cyclic. It remains to prove that every p_1-Sylow subgroup of G is cyclic.
For, let H be a p_1-Sylow subgroup of G. Thus, $G = C_m \rtimes_\varphi H$, for some $\varphi : H \to \mathrm{Aut}(C_m)$. Since $G/C_m = H$, by Lemma 2.1, either H is simple, or H is an NCS group. If H is simple, then $H = C_{p_1}$. If H is not simple, then, by Proposition 3.1 and Proposition 3.3, there are three possibilities for H: either is cyclic of order $p_1^{\alpha_1}$,

or is Q_8, or is $C_{p_1} \times C_{p_1}$. We claim that the only possibility here is that $H = C_{p_1^{\alpha_1}}$, for some $\alpha_1 > 0$.
Aiming for a contradiction, suppose that $H = Q_8$. Since $C_4 \trianglelefteq Q_8$, $C_m \rtimes_\psi C_4 \trianglelefteq C_m \rtimes_\varphi Q_8 = G$, where ψ is the restriction of φ to C_4. Recall that Q_8 has 3 normal subgroups isomorphic to C_4, and their union covers Q_8. Since G is an NCS group, we have that

$$C_m \rtimes_\psi C_4 = C_{4m} = C_4 \times C_m,$$

since $(m, 4) = 1$. This holds for each of the three copies of C_4 inside Q_8, implying that ψ is the identity for each C_4, so that φ is also the identity. But then $G = C_m \times Q_8$, and this is a contradiction, since Q_8 is not cyclic.
In a very similar way, we can show that H cannot be $C_p \times C_p$, and this concludes the proof. □

Remark. If G is a Z-group, it is not necessarily an NCS group. The smallest example is the group of order 20 with presentation

$$G = \langle a, b \,|\, a^5 = b^4 = 1, bab^{-1} = a^4 \rangle.$$

This is the group with identifier $(20, 3)$ in the Small Groups library in GAP [7], and it is easy to see that this has a normal subgroup isomorphic to D_{10}, which is not cyclic.

With this in hand, we can state the following proposition, which classifies all NCS groups with proper commutator subgroup, which are not p-groups. Before, we need a very basic number theoretic lemma.

Lemma 3.5. *Let m, n, r be three integers. Suppose that $(m, n) = (m, r-1) = 1$ and that $r^{n_1} \equiv 1 \mod m$ for any $n_1 | n$, with $n_1 \neq 1$. Then, $n = p^\alpha$ for some prime p.*

Proof. Let d be the order of r modulo m, that is, the smallest integer such that $r^d \equiv 1 \mod m$. Since $(m, r-1) = 1$, $d > 1$. Let $n_1 \neq 1$ be a divisor of n. Since $r^{n_1} \equiv 1 \mod m$, we have that $d | n_1$. So, n has a divisor which divides every other divisor of n, and this can happen if and only if $n = p^\alpha$ for some prime p. □

Proposition 3.6. *Let G be a* Z*-group with presentation*

$$G = \langle a, b \,|\, a^m = b^n = 1, bab^{-1} = a^r \rangle,$$

with $(m, n) = 1$, $(m, r-1) = 1$ and $r^n \equiv 1 \mod m$. Then, G is an NCS *group if and only if $n = p^\alpha$, for some prime p such that $p < q$ for any prime q dividing m, and $r^p \equiv 1 \mod m$.*

Proof. Observe that $|G'| = m$, and $|G/G'| = n$. Suppose first that G is an NCS group, and let

$$N = \langle a, b^{n_1} \rangle \trianglelefteq G,$$

for some divisor $n_1 | n$, $n_1 \neq 1$. Since N is cyclic, its generators commute, that is

$$b^{n_1} a b^{-n_1} = a.$$

An easy computation shows that

$$b^{n_1} a b^{-n_1} = a^{r_1^n}.$$

In conclusion, we have $a^{r^{n_1}} = a$, that is $a^{r^{n_1} - 1} = 1$, and this happens if and only if $r^{n_1} \equiv 1 \mod m$. By Lemma 3.5, we have that $n = p^\alpha$ for some prime p. Let

$|G| = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$, where $p_1 < p_2 < \dots p_t$ are primes, α_i are non-negative integers and $t > 1$. Since G is supersolvable, G admits a normal subgroup N of order $p_2^{\alpha_2} \cdots p_t^{\alpha_t}$. We have showed, in the proof of Proposition 3.4, that G/N is abelian. In particular, $G' \leq N$. Now, we have that $nm = p^\alpha |G'| = p_1^{\alpha_1}|N|$. Therefore, $p_1 | p^\alpha |G'|$. However, $G' \leq N$, and therefore $p_1 | p^\alpha$, and thus $p_1 = p$ and $G' = N$.
Suppose now that $n = p^\alpha$ for some prime p less then all the prime divisors of m, and that $r^p \equiv 1 \mod m$. Take N to be a normal subgroup of G. By Theorem 2.5, N is generated by

$$N = \langle a^{m_1}, b^{p^\beta} \rangle,$$

for some $m_1 | (m, r^{p^\beta} - 1)$, and some $\beta \leq \alpha$. An easy computation shows that

$$b^{p^\beta} a^{m_1} b^{-p^\beta} = a^{m_1 r^{p^\beta}}, \tag{1}$$

hence $\langle a^{m_1} \rangle \trianglelefteq N$. In particular, N is the semidirect product of the cyclic groups generated by a^{m_1} and b^{p^β} respectively. Thus, N is cyclic if and only if

$$b^{p^\beta} a^{m_1} b^{-p^\beta} = a^{m_1}.$$

so that, by (1), N is cyclic if and only if

$$a^{m_1(r^{p^\beta} - 1)} = 1.$$

Now $r^p \equiv 1 \mod m$, so that for every β we have $r^{p^\beta} \equiv 1 \mod m$, and thus

$$a^{m_1(r^{p^\beta} - 1)} = 1.$$

In conclusion, N is cyclic. Since N is arbitrary, G is an NCS group. □

Remark. In the example above, $m = 5, p = 2$ and $r = 2$. In this case, r^p is not congruent to 1 modulo 5, hence, as we have seen, the group is not an NCS group. However, if we take $r = -1$, then $r^p \equiv 1 \mod 5$, and $G = \langle a, b \,|\, a^5 = b^4 = 1, bab^{-1} = a^{-1} \rangle$ is an NCS group.

4. Proof of Theorem 1.1: Case $G' = G$

In this case, G is perfect. We require the following known lemma, of which we report the very short proof.

Lemma 4.1. *Let G be a perfect group and let N be a cyclic normal subgroup of G. Then, $N \leq Z(G)$.*

Proof. Since N is normal in G, we have $N_G(N)/C_G(N) = G/C_G(N) \leq \mathrm{Aut}(N)$. N is cyclic, so its automorphism group is abelian. Therefore, $G/C_G(N)$ is abelian, implying that $C_G(N) \geq G' = G$, so that $C_G(N) = G$, which yields $N \leq Z(G)$. □

Lemma 4.2. *Let G be a perfect* NCS *group. Then, G is quasisimple.*

Proof. If $N \trianglelefteq G$, then N is cyclic. By Lemma 4.1, $N \leq Z(G)$, so that $Z(G)$ is a maximal normal subgroup of G. □

To continue our analysis, we need some basic results about perfect central extensions and the Schur multiplier.
Recall that a central extension of a group G is a pair (H, α) such that we have the following short exact sequence:

$$1 \to Z \to H \xrightarrow{\alpha} G \to 1,$$

where $Z \leq Z(H)$. If H is a perfect group, then the central extension is said to be perfect.
If G is a perfect group, then G admits a special central extension, called *universal central extension* of G, denoted with $(\tilde{G}, \pi)$. The kernel of π is called the *Schur multiplier* of G, and it is denoted by $H^2(G, \mathbb{Z})$. This extension has the property that for any other perfect central extension of G, say (H, φ), $\ker \varphi$ is a quotient of $H^2(G, \mathbb{Z})$. We refer the reader to [2] for a detailed discussion of this material.

Proposition 4.3. *Let G be a perfect group. Then, G is an* NCS *group if, and only if, G has a unique maximal normal subgroup and it fits into a short exact sequence of the form*
$$1 \to C \to G \to S \to 1,$$
where S is a simple group, and C is a cyclic group isomorphic to a quotient of the Schur multiplier of S.

Proof. Suppose that G is an NCS perfect group. Then, By Lemma 4.2, $Z(G)$ is the unique maximal normal subgroup of G, and G fits into a short exact sequence of the form
$$1 \to Z(G) \to G \to S \to 1,$$
where $S = G/Z(G)$ is simple. Moreover, S is also perfect, since
$$S' = \left(\frac{G}{Z(G)}\right)' = \frac{Z(G)G'}{Z(G)} = \frac{G}{Z(G)} = S.$$
Let $(\tilde{S}, \pi)$ be the universal central extension of S. Thus, we have an exact sequence of the form
$$1 \to H^2(S, \mathbb{Z}) \to \tilde{S} \to S \to 1.$$
But, since also G is a perfect central extension of S, we have that
$$Z(G) \cong \frac{H^2(S, \mathbb{Z})}{N},$$
for some $N \trianglelefteq H^2(S, \mathbb{Z})$.
Suppose now that G fits into a short exact sequence of the form
$$1 \to C \to G \to S \to 1,$$
where S is simple, and C is cyclic a quotient of the Schur multiplier of S. Since G/C is simple, C is the unique maximal normal subgroup. Therefore, every normal subgroup of G is contained in C, and hence it is cyclic. □

Let G be a perfect NCS group. Then, G fits into a short exact sequence
$$1 \to C \to G \to S \to 1,$$
where S is a simple group, and C is a quotient of the Schur multiplier of S.
By the Classification Theorem of Finite Simple Groups, we can list all possibilities of the couple (C, S): these are reported in Table 1.
First of all, we have to exclude all simple groups with trivial Schur multiplier.
If S is a simple group with cyclic Schur multiplier, we can take as C all the possible quotients of the Schur multiplier.
Thus, it remains to work with the simple groups with non-cyclic non-trivial Schur multiplier, which are reported in Table 2 (we use [11] to see the Schur multiplier of all simple groups).

S	$H^2(S,\mathbb{Z})$	Comments
$\mathrm{PSL}_3(4)$	$C_{12} \times C_4$	
$\Omega^+_{2n}(q)$	$C_2 \times C_2$	n even, $(4, q^n - 1) = 4$
$\Omega^+_8(2)$	$C_2 \times C_2$	
$\mathrm{PSU}_4(3)$	$C_{12} \times C_3$	
$\mathrm{PSU}_6(2)$	$C_6 \times C_2$	
${}^2E_6(2^2)$	$C_6 \times C_2$	
${}^2B_2(8)$	$C_2 \times C_2$	

Table 2. Finite simple groups with non-trivial non-cyclic Schur multiplier

We now illustrate with an example how Table 1 is constructed for these groups. Suppose that G is an NCS perfect central extension of the simple group $S = \mathrm{PSL}_3(4)$. Thus, G fits in a short exact sequence of the form

$$1 \to \frac{C_{12} \times C_4}{N} \hookrightarrow G \to \mathrm{PSL}_3(4) \to 1,$$

for $N \trianglelefteq C_{12} \times C_4$ and $(C_{12} \times C_4)/N = Z(G)$. Since G is an NCS group, and every normal subgroup of G is a subgroup of $(C_{12} \times C_4)/N$, this last group has to be an NCS group. So we need to consider all NCS quotients of $C_{12} \times C_4$. In particular, such a quotient must be cyclic. In conclusion, all the possible choices for N are the following (up to isomorphism):

$$C_{12},\ C_4,\ C_4 \times C_4,\ C_4 \times C_2.$$

Summing up, the NCS perfect groups which are a central extension of $S = \mathrm{PSL}_3(4)$ are extension by S of one of the following cyclic groups

$$C_4,\ C_{12},\ C_3,\ C_6.$$

For all the other groups in Table 2, Theorem 1.1 is proved in a similar fashion.

In Table 1, all simple groups S with non-trivial Schur multiplier are reported in the first column. In the second column, there are all possible cyclic groups with which we can extend the group S and obtain an NCS group.

References

[1] S. M. Alshorm, Groups in which the commutator subgroup is cyclic, *International Conference on Mathematics and Computations*, 2022.
[2] M. Aschbacher, Finite group theory, *Cambridge University Press* Vol. 10, 2000.
[3] A. Ballester-Bolinches, R. Esteban-Romero, On minimal non-supersoluble groups, *Revista Matemática Iberoamericana*, vol. 23, no.1, 2007, pp. 127-142.
[4] A. Ballester-Bolinches, R. Esteban-Romero, D. J. S. Robinson, On Finite Minimal Non-Nilpotent Groups, *Proceedings of the American Mathematical Society*, vol. 133, no. 12, 2005, pp. 3455–62.
[5] M. Bianchi, Sui gruppi a sottogruppi normali ciclici, *Istit. Lombardo Accad. Sci. Lett. Rend. A*, no. 112(2), 1978, pp. 302–310.
[6] K. Conrad, Subgroup series II, `https://kconrad.math.uconn.edu/blurbs/grouptheory/subgpseries2.pdf`
[7] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.13.1*; 2024, `https://www.gap-system.org`.
[8] X. Guo, Q. Li, On generalization of minimal non-nilpotent groups, *Lobachevskii J Math* 31, 239–243 (2010)
[9] M. Hall Jr, The theory of groups, 1959

[10] G. A. Miller, and H. C. Moreno, Non-Abelian Groups in Which Every Subgroup Is Abelian, *Transactions of the American Mathematical Society*, vol. 4, no. 4, 1903, pp. 398–404.
[11] R. Wilson, The finite simple groups, *London: Springer*, vol. 251, 2009.
[12] W. R. Scott, Group Theory, Dover Books on Mathematics, Dover Publications, 1987.
[13] M. Suzuki, On finite groups with cyclic Sylow subgroups for all odd primes, *American journal of mathematics* 1955, pp. 657-691.
[14] M. Tărnăuceanu, The normal subgroup structure of ZM-groups, *Annali di Matematica* 193, 1085–1088 (2014).

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